AN INVESTIGATION OF THE COUPLING BETWEEN LONGITUDINAL AND FLEXURAL MODES OF VIBRATION CAUSED BY AN OFFSET OF THE CENTER OF GRAVITY FROM THE CENTER OF STIFFNESS

by

RICHARD F. BURNS, LIEUTENANT, U.S. NAVY

B.S., UNITED STATES NAVAL ACADEMY
(1954)

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

NAVAL ENGINEER

AND THE DEGREE OF

MASTER OF SCIENCE IN NAVAL ARCHITECTURE

AND MARINE ENGINEERING

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1962

Professor Frank M. Lewis

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by

Richard F. Burns, Lt. USN

Submitted to the Department of Naval Architecture and Marine Engineering on 18 May 1962 in partial fulfillment of the requirements for the degree of Naval Engineer and the degree of Master of Science in Naval Architecture and Marine Engineering.

ABSTRACT

The purpose of the investigation is to find how the natural frequency of transverse vibration is affected when the center of gravity and center of stiffness are not coincident and to determine if it is important in calculating the natural frequency of the fundamental mode of vibration of ships.

Two methods, one an approximate analytic method, and the other a lumped mass method, are used to determine the strength of the coupling of these modes of vibration. The approximate analytic method involves the solution of differential equations from an assumed mode shape and then the use of Stodola's equation to compute the change in natural frequency. The lumped mass method is a numerical procedure requiring a considerable amount of computation. Both methods are in good agreement when the ratio of natural transverse to longitudinal frequency is less than one-half.

The approximate analytic method shows that the change in transverse frequency of vibration is a function of the product of two factors. One factor is directly proportional to the area times the distance of the center of gravity from the center of stiffness squared and inversely proportional to the area moment of inertia. The other factor is a complicated expression, but which is dependent only on the ratio of transverse to longitudinal frequency as computed by simpler theory in beams, neglecting the offset of the center of gravity from the center of stiffness. These expressions have been evaluated and plotted against frequency ratio. A simple method of computing the change in transverse frequency of vibration is provided.

When the important parameters for calculating the change in frequency of vibration are calculated for a merchant ship and submarine, it is seen that the coupling of the fundamental modes of vibration in these structures is insignificant.

In the higher modes of vibration, the ratio of transverse to longitudinal frequencies approaches unity. It is recommended that the coupling in these higher modes be investigated.

Thesis Supervisor: Frank M. Lewis

Title: Professor of Marine Engineering

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NOTATION

A	area of a cross section which will support longitudinal stress
c	arbitrary constant
E	modulus of elasticity
f	ratio of transverse to longitudinal frequency computed disregarding the offset of the center of gravity from the center of stiffness
F	force
g	gravity
I	area moment of inertia of a cross section
K	spring constant
L	length of a beam
ℓ	distance between masses in the lumped mass method
M	moment
m	mass
S	stress
u	longitudinal displacement
V	shear
w	frequency
$w_{o\ell}$	longitudinal frequency as computed disregarding the offset of the center of gravity from the center of stiffness.
w_{ot}	transverse frequency as computed disregarding the offset of the center of gravity from the center of stiffness.
w_t	transverse frequency with coupled longitudinal motion
y	vertical displacement
$\bar{y}$	distance of the center of gravity from the center of stiffness
θ	slope
λL	a number determined by the transverse mode of vibration. It equals 4.73 for the fundamental mode.
(αA)	weight per unit length

INTRODUCTION

The calculation of the natural frequencies of the longitudinal and transverse modes of vibration of any structure have long been considered independent.[1] In a uniform bar, of course, it is true, but for example, in a ship's hull where the center of mass and center of stiffness are not coincident, it is not necessarily true. A bending of the ship's structure causes some motion in the longitudinal direction of the center of gravity. This, in turn, will cause some change in the natural frequency of transverse vibration. It is the purpose of this thesis to investigate the strength of the coupling of longitudinal and transverse modes of vibration when the centers of gravity and stiffness are displaced and determine if it has any significance in the calculation of natural frequencies of ships.

PROCEDURE

Two distinct procedures are used to find the coupling of longitudinal and flexural modes of vibration; one method is an approximate analytic method and the other is a lumped mass method. The steps to each method are outlined in subsections A and B below; the details are in the appendices. Through a numerical example the results of the two procedures are then compared for agreement. The entire analysis is then interpreted in the light of typical values of significant parameters for a submarine and a merchant ship.

The investigation is restricted to the fundamental mode of vibration.

In neither the approximate analytic method nor the lumped mass method is the deflection of the system due to shear and rotary inertia considered. The transverse motion is due to bending only.

A. Approximate Analytic Method.

The general procedure in this method is to find a correction to an assumed mode shape from the differential equations of a system in which bending causes longitudinal motion and longitudinal forces cause bending. The change in mode shape is then used as a measure of the change in natural frequency of transverse vibration. The steps in the procedure which are carried out in Appendix I are:

1. Determine the differential equations of motion of a system in which the center of gravity and center of stiffness are not coincident. The first equation is a longitudinal force balance equation and the second one is a bending moment equation.

$$- \frac{(\alpha A)}{g} \frac{\partial^2}{dt^2} \left(u + \frac{\partial y}{\partial x} \bar{y}\right) + AE \frac{d^2 u}{dx^2} = 0 \qquad (1)$$

$$\frac{(\alpha A)}{g} \frac{\partial^2 y}{dt^2} + EI \frac{\partial^4 y}{\partial x^4} + EA \frac{\partial^3 u}{\partial x^3} \bar{y} = 0 \tag{2}$$

(αA) = weight per unit length

A = area of cross section

g = gravity

u = longitudinal displacement

y = vertical displacement

E = Young's modulus

$\bar{y}$ = distance of center of gravity from the center of stiffness

I = area moment of inertia of a cross section

2) Assume a transverse mode shape of a free-free beam in space.

$$y = c\left[\cos \lambda x + \left(\frac{\text{Cos } \lambda L/2}{\text{Cosh } \lambda L/2}\right) \text{Cosh } \lambda x\right] \tag{3}$$

c = arbitrary constant

λL = a number determined by the mode of vibration. It is equal to 4.73 for the fundamental mode of vibration.

L = length of the structure.

3) From the differential equation for longitudinal force balance, find the longitudinal motion of the center of stiffness produced by the mode shape.

4) This longitudinal motion causes longitudinal forces which produce moments about the center of gravity and hence additional bending of the system. Determine the additional deflection from these forces using a bending moment equation.

5) With the results of Step 4 recycle Steps 3 and 4. Hence, a first and a second approximation to the additional deflection can be obtained.

6) Use a method similar to that of Stodola to find the transverse frequency with coupled longitudinal motion. The ratio of the square of the transverse frequency with longitudinal motion to that without is proportional to the ratio of total deflection without longitudinal motion to the total deflection of the beam with longitudinal motion.

B. Lumped Mass Method.

The lumped mass procedure is the Prohl-Myklestad sequence method for calculating the natural frequency of beams of variable cross section. This calculation is modified to include the effect of the displacement of the center of gravity from the center of stiffness.

The calculation is based on the fundamental equations of bending and strain in beams:

$$\frac{M}{EI} = \frac{d^2y}{dx^2} \qquad \text{and}$$

$$F = \frac{(AE)}{\ell} u$$

The beam is divided into sections. At each section an inertia loading is added:

in shear equal to

$$m\, w^2\, y$$

in bending equal to

$$m\, w^2\, \bar{y}\, (u - \bar{y}\, \theta)$$

and in longitudinal force equal to

$$m\, w^2\, (u - \bar{y}\, \theta)$$

m = mass

w = frequency

θ = slope

ℓ = length

The procedure is to calculate the longitudinal force and displacement, transverse deflection, slope, bending moment and shear at each section along the beam in proceeding from the end of the beam to the mid-point. Various frequencies are tried until the boundary conditions are satisfied. For the fundamental mode, the slope, shear, and longitudinal motion at the mid-point are zero.

RESULTS

A. Approximate Analytic Method

The equation for determining the frequency of transverse vibration by the approximate analytic method is:

$$w_t^2 = w_{ot}^2 \frac{Y_o}{Y_o + Y_1 + Y_2} \tag{4}$$

where:

$$w_{ot}^2 = \frac{(22.4\ )^2}{L^4} \frac{EgI}{(\alpha A)} \tag{5}$$

$$Y_o = c\left[2 \text{ Cos } (\lambda L/2) - 1 - \frac{\text{Cos } (\lambda L/2)}{\text{Cosh } (\lambda L/2)}\right] \tag{6}$$

$$Y_1 = \frac{A}{I} \bar{y}^2 c \text{ Cos } \lambda L/2 \left[\frac{-2 D_1^2 \lambda^4}{1 - D_1^2 \lambda^4} \frac{(\text{Cos} \frac{L/2}{\sqrt{D_1}} - 1)}{\text{Cos} \frac{L/2}{\sqrt{D_1}}}\right.$$

$$\left. + \frac{(\text{Cos } \lambda L/2 - 1)}{(1 - D_1 \lambda^2) \text{ Cos } \lambda L/2} + \frac{1}{(1 + D_1\lambda^2)} \frac{(\text{Cosh } \lambda L/2 - 1)}{\text{Cosh } \lambda L/2}\right] \tag{7}$$

$$Y_2 = \left(\frac{A}{I}\right)^2 \bar{y}^4 c \text{ Cos } \lambda L/2 \left[\left(\frac{-2 D_1^2 \lambda^4}{1 - D_1^2 \lambda^4} - \frac{4 D_1^2 \lambda^4}{(1 - D_1^2 \lambda^4)^2}\right)\cdots\right.$$

$$\cdots\left(\frac{\text{Cos}\frac{L/2}{\sqrt{D_1}} - 1}{\text{Cos}\frac{L/2}{\sqrt{D_1}}}\right) - \left(\frac{D_1^2 \lambda^4}{1 - D_1^2 \lambda^4}\right) \left(\frac{L/2}{\sqrt{D_1}}\right) \frac{\text{Sin}\frac{L/2}{\sqrt{D_1}}}{(\text{Cos}\frac{L/2}{\sqrt{D_1}})^2}$$

$$+ \frac{1}{(1 - D_1 \lambda^2)^2} \frac{(\text{Cos } \lambda L/2 - 1)}{\text{Cos } \lambda L/2} + \frac{1}{(1 + D_1 \lambda^2)^2}\cdots$$

$$\left.\cdots \frac{(\text{Cosh } \lambda L/2 - 1)}{\text{Cosh } \lambda L/2}\right] \tag{8}$$

w_t =transverse frequency with coupled longitudinal motion

w_{ot} =transverse frequency as calculated disregarding the offset of the center of gravity from the center of stiffness

Y_o =total deflection of the assumed mode shape

Y_1 = first approximation to the additional deflection

Y_2 = second approximation to the additional deflection

When substitutions shown in Appendix A are made, all the terms in brackets in equations (7) and (8) are, to an approximation, dependent only on frequency ratio, f.

$$f = \frac{w_{ot}}{w_{o\ell}}$$

where

$w_{o\ell}$ = longitudinal frequency as calculated disregarding the offset of the center of gravity from the center of stiffness. Specifically,

$$D_1 \lambda^2 = \left(\frac{w_{o\ell}}{w_t}\right)^2 \left(\frac{4.73}{\pi}\right)^2$$

If $w_t \approx w_{ot}$

$$D_1 \lambda^2 = \left(\frac{4.73}{f\pi}\right)^2 = \left(\frac{\lambda L}{f\pi}\right)^2$$

Similarly,

$$\frac{L/2}{\sqrt{D_1}} = f\,\frac{\pi}{2}$$

Another important result derived in Appendix A is that the frequency ratio, f, is independent of mass per unit length.

These two important results suggest a rearrangement of the frequency equation 4, to permit easy calculation of the change in frequency for any system.

$$\left(\frac{w_t}{w_{ot}}\right)^2 = \frac{1}{1 + \frac{Y_1}{Y_o} + \frac{Y_2}{Y_o}}$$

This equation can be put in a still more usable form.

$$\left(\frac{w_t}{w_{ot}}\right)^2 \quad \frac{1}{1 + \frac{A\bar{y}^2}{I}\, g(f) + \left(\frac{A\bar{y}^2}{I}\right)^2 h(f)} \tag{9}$$

where g(f) and h(f), functions which are only dependent on frequency ratio, can be obtained from Figures I and II.

The function for the first approximation to beam deflection is:

$$g(f) = \frac{\operatorname{Cos}\frac{\lambda L}{2}\left[\frac{-2\left(\frac{\lambda L}{f\pi}\right)^4}{\left(1 - \left(\frac{1}{f}\frac{\lambda L}{\pi}\right)^4\right)}\frac{(\operatorname{Cos} f\frac{\pi}{2} - 1)}{\operatorname{Cos}(f\frac{\pi}{2})} + \frac{(\operatorname{Cos}\lambda L/2 - 1)}{\left(1 - \left(\frac{\lambda L}{f\pi}\right)^2\right)\operatorname{Cos}\frac{\lambda L}{2}} + \frac{(\operatorname{Cos}\lambda L/2 - 1)}{\left(1 + \left(\frac{\lambda L}{f\pi}\right)^2\right)(\operatorname{Cosh}\lambda L/2)}\right]}{\left[2\operatorname{Cos}\lambda L/2 - 1 - \frac{\operatorname{Cos}\lambda L/2}{\operatorname{Cosh}\lambda L/2}\right]}$$

The function for the second approximation to beam deflection is:

$$h(f) = \operatorname{Cos}\lambda L/2\left[\left(\frac{-2\left(\frac{\lambda L}{f\pi}\right)^4}{1 - \left(\frac{\lambda L}{f\pi}\right)^4} - \frac{4\left(\frac{\lambda L}{f\pi}\right)^4}{\left(1 - \left(\frac{\lambda L}{f\pi}\right)^4\right)^2}\right)\left(\frac{\operatorname{Cos} f\frac{\pi}{2} - 1}{\operatorname{Cos} f\frac{\pi}{2}}\right) - \left(\frac{\left(\frac{\lambda L}{f\pi}\right)^2}{1 - \left(\frac{\lambda L}{f\pi}\right)^2}\right)\cdots\right.$$

$$\cdots\left(f\frac{\pi}{2}\right)\frac{\operatorname{Sin} f\frac{\pi}{2}}{(\operatorname{Cos} f\frac{\pi}{2})^2} + \frac{(\operatorname{Cos}\lambda L/2 - 1)}{\left(1 - \left(\frac{\lambda L}{f\pi}\right)^2\right)^2\operatorname{Cos}\lambda L/2}$$

$$\left. + \frac{(\operatorname{Cosh}\lambda L/2 - 1)}{\left(1 + \left(\frac{\lambda L}{f\pi}\right)^2\right)^2\operatorname{Cosh}\lambda L/2}\right] \div \left[2\operatorname{Cos}\lambda L/2 - 1\right.$$

$$\left. - \frac{\operatorname{Cos}\lambda L/2}{\operatorname{Cosh}\lambda L/2}\right]$$

These expressions do not have to be evaluated in finding the change in natural frequency. All that has to be evaluated is w_{ot}, $w_{o\ell}$, and $\frac{A\bar{y}^2}{I}$ with g(f) and h(f) from Figures I and II to use the approximate analytic method.

Above a frequency ratio of one-half the approximate analytic method gives poor results. However, it is still desirable to estimate frequency changes at higher frequency ratios. An examination of the relative order of magnitude of these two functions suggests an alternate method of determining frequency changes from data of only one numerical example of the lumped mass

FIGURE I

FUNCTION FOR THE FIRST APPROXIMATION TO THE CHANGE IN BEAM DEFLECTION

g(f) versus f

f	1/4	3/8	1/2	5/8	3/4
g(f)	.0655	.1609	.3275	.6316	1.2675

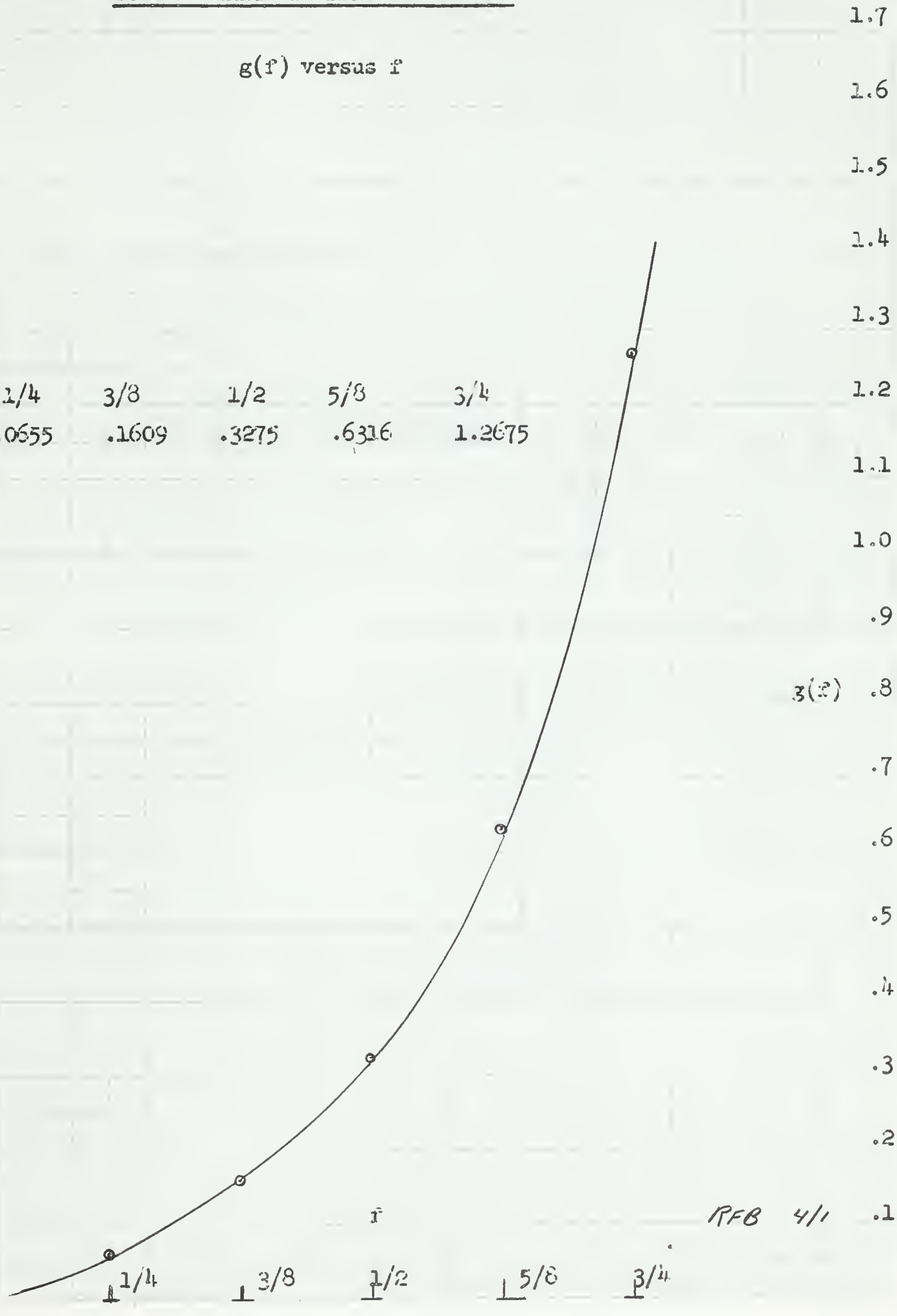
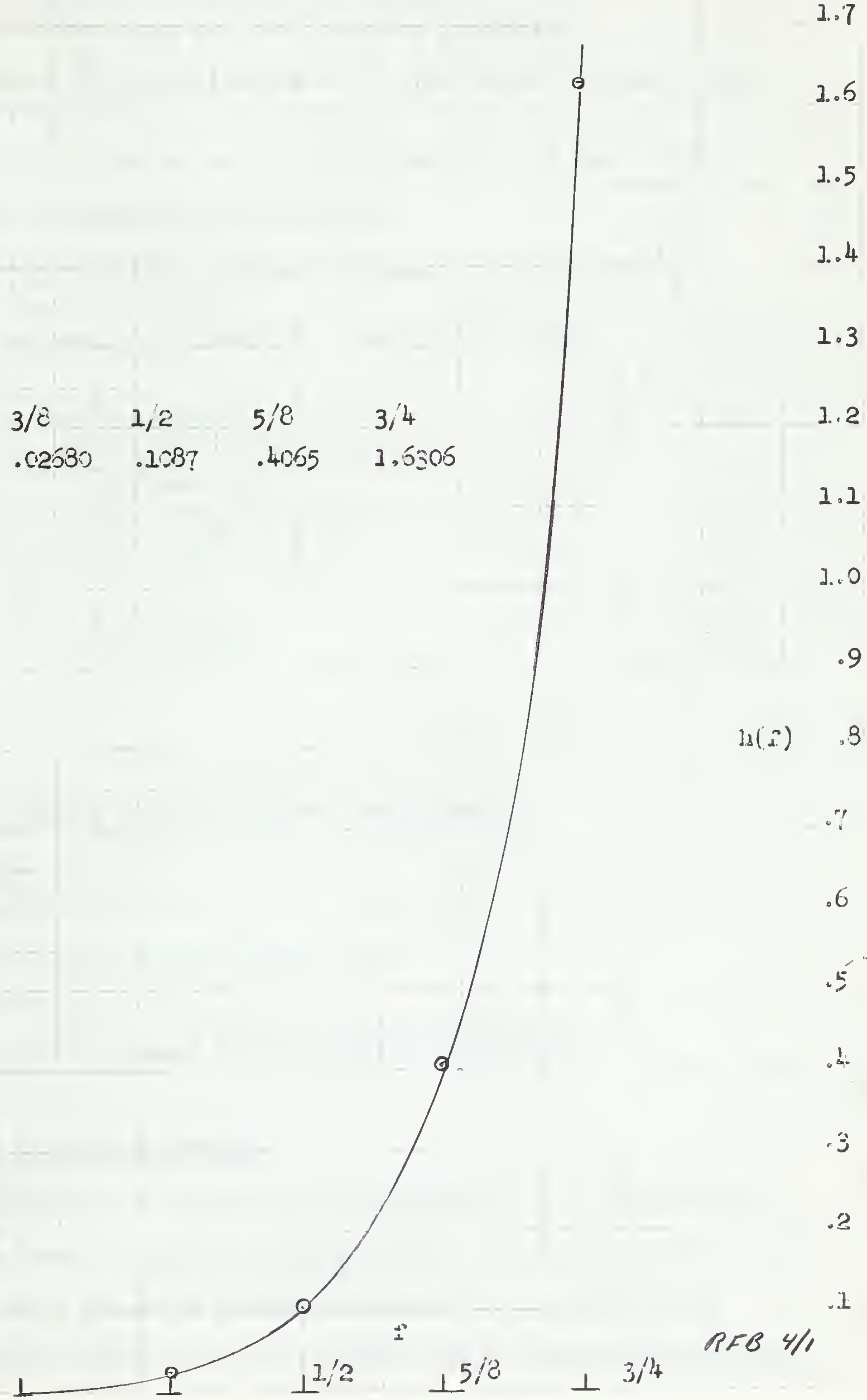


FIGURE II

FUNCTION FOR THE SECOND APPROXIMATION

TO THE CHANGE IN BEAM DEFLECTION

f	1/4	3/8	1/2	5/8	3/4
h(f)	.0043	.02680	.1087	.4065	1.6306



method shown in Figure IV. This alternate method is based on the fact that

$$h(f) \approx \left[g(f)\right]^2$$

To use Figure IV for other beams use the following procedure:

1. Determine $\frac{w_t}{w_{ot}}$ from this figure for the desired frequency ratio.
2. This number can be used to solve Equation (10) for x. The solution is shown as Equation (11).
3. Use this value of x in a new frequency equation with $\frac{A\bar{y}^2}{I}$ for the desired application. See Equation (12).

$$\left(\frac{w_t}{w_{ot}}\right)^2 = \frac{1}{1 + a_o x + a_o^2 x^2} \qquad (10)$$

$$x = \frac{1}{a_o}\left[-\frac{1}{2} + \sqrt{\frac{1}{4} + \left(\frac{w_{ot}}{w_t}\right)^2 - 1}\right] \qquad (11)$$

$$\left(\frac{w_t'}{w_{ot}}\right)^2 = \frac{1}{1 + a_1 x^2 + a_1^2 x^2} \qquad (12)$$

where

$\frac{w_t}{w_{ot}}$ = frequency change obtained from Figure IV

$a_o = \frac{A\bar{y}^2}{I} = .75$

$a_1 = \frac{A\bar{y}^2}{I}$ for the desired application

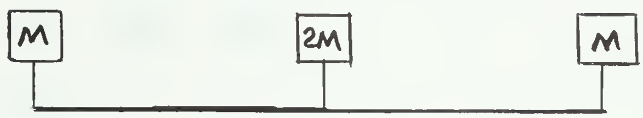
$\frac{w_t'}{w_{ot}}$ = frequency change for the desired application.

B. Results of the Lumped Mass Method.

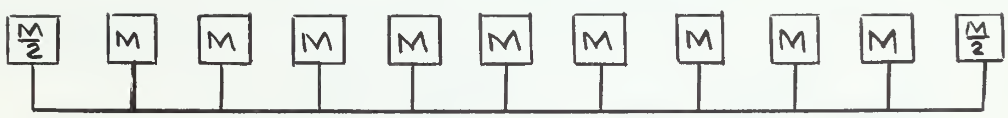
The equations for determining the conditions at each mass along the beam are shown below. The zero subscript refers to the conditions at the previous section; the 1 subscript refers to the mass at the point being evaluated. The actual numerical value of each of these equations becomes the

FIGURE III

LUMPED MASS SYSTEMS



THREE-MASS SYSTEM



ELEVEN-MASS SYSTEM

subscript zero term when the next point along the beam is evaluated.

$$M_1 = M_o + V_o \ell + \bar{y}\, m_1\, w^2\, (u_1 - \bar{y}\, \theta_1)$$

$$V_1 = V_o + m_1\, w^2\, y_1$$

$$\theta_1 = \theta_o + \frac{M_o \ell}{EI} + \frac{V_o \ell^2}{2EI}$$

$$y_1 = y_o + \theta_o \ell + \frac{M_o \ell^2}{2EI} + \frac{V_o \ell^3}{6EI}$$

$$u_1 = u_o - \frac{F_o}{K}$$

$$F_1 = F_o + m_1\, w^2\, (u_1 - \bar{y}\, \theta_1)$$

where

M = bending moment

V = shear force

F = longitudinal force

K = spring constant between masses

ℓ = distance between masses

One advantage of the lumped mass method is that it is useful in deriving a frequency equation. For the three mass systems shown in Figure III, the frequency equation is:

$$0 + 2 - m\, w^2 \left(\frac{1}{3}\frac{\ell^3}{EI} + \frac{2\, \ell\, \bar{y}^2}{EI} + \frac{2}{K}\right) + m^2\, w^4 \left(\frac{\ell^4\, \bar{y}^2}{12EI} + \frac{1}{3}\frac{\ell^3}{EIK}\right)$$

C. <u>Results of a Numerical Example.</u>

In order to check the validity of the results, a numerical example was done using both methods. The change in frequency is shown plotted in Figure IV as a function of frequency ratio, f. The plot is for a system of these properties:

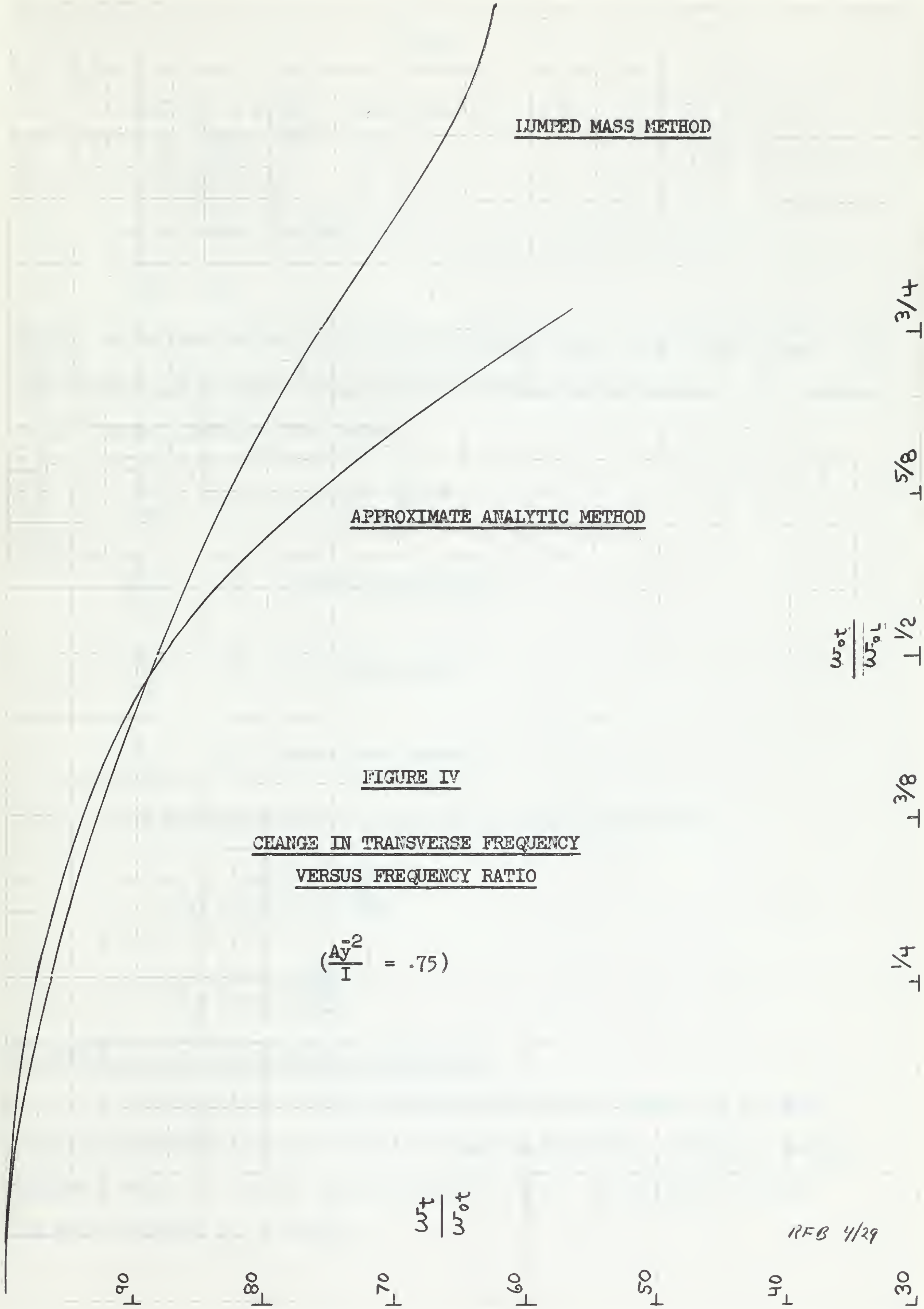


FIGURE IV

CHANGE IN TRANSVERSE FREQUENCY

VERSUS FREQUENCY RATIO

$\left(\frac{A\bar{y}^2}{I} = .75\right)$

$A = 1 \quad in^2$

$I = 1/12 \quad in^4$

$\bar{y} = 1/4 \quad in$

$\alpha = .284 \quad lbs/in^3$

$\frac{A\bar{y}^2}{I} = .75$

In the lumped mass method the system was divided into eleven masses (Figure III). For system of fewer masses the agreement between methods was poor. For example, at a frequency ratio of one-fourth:

$\frac{w_t}{w_{ot}} = .974$ analytic method

$\frac{w_t}{w_{ot}} = .961$ eleven mass system

$\frac{w_t}{w_{ot}} = .950$ nine mass system

$\frac{w_t}{w_{ot}} = .910$ seven mass system

For all numerical work, w_{ot} and $w_{o\ell}$ were computed from:

$$w_{ot} = \frac{22.4}{L^2} \sqrt{\frac{gEI}{(\alpha A)}}$$

$$w_{o\ell} = \frac{\pi}{L} \sqrt{\frac{EgA}{(\alpha A)}}$$

D. Numerical values for Marine Application.

The approximate analytic method indicates that there are two significant parameters in calculating the change in transverse frequency, namely, frequency ratio, f, and the geometric factor $\frac{A\bar{y}^2}{I}$. The geometric factor has been evaluated for two ships:

merchant ship (3) $\frac{Ay^2}{I} \approx .087$

submarine (4) $\frac{Ay^2}{I} \approx .017$

These numbers, of course, are dependent on the condition of loading and, furthermore, they are valid only for the midship section. The geometric factor for the merchant represents an upper limit of this number for the "Gopher Mariner". Approximate frequency ratios for these ship types are:

submarine (4) $f \approx .15$

merchant ship $f \approx .25$

DISCUSSION OF RESULTS, CONCLUSIONS, AND RECOMMENDATIONS

A. Accuracy of Methods

The approximate analytic method is inaccurate above the frequency ratio of one-half. The lumped mass method, on the other hand, can be used over the entire range of frequency ratio.

Reference (5) states that to ensure better than one percent accuracy in computing frequencies by the lumped mass method, thirteen masses should be used per wave length. If more masses had been used in the numerical example, the agreement between methods at the low frequency ratio would be better than it already is. For eleven masses there is only a 1.3% difference.

B. Significance in Ship Vibrations.

When the constants for the submarine and merchant ship stated in Part III above are used in Equation (9) in conjunction with Figures I and II, the change in transverse frequency is truly insignificant for the fundamental mode. For higher modes of vibration, the frequency ratio is larger and the change in frequency might not be so insignificant.

C. General

Two general methods for calculating frequency change due to an offset of the center of gravity from the center of stiffness are given . One method is approximate; the other is exact. Also, a method of estimating frequency change by a simple calculation from one lumped mass numerical example is explained. The coupling of the transverse and longitudinal modes of vibration in any structure can be estimated or obtained exactly by the methods explained in the text of this thesis.

All the work is restricted to the fundamental mode of vibration. The lumped mass method, of course, could be used for higher modes. It is recommended that the coupling due to the offset of the center of gravity from the center of stiffness for higher modes be investigated.

APPENDIX

APPENDIX A

DETAILS OF THE APPROXIMATE ANALYTIC METHOD

The approximate analytic method, outlined below, is divided into steps, the numbering of which corresponds to the numbering in the procedure Section IIA above. The description which appears in that section is not repeated.

Step 1: Derivation of the Equations of Motion

With the sign convention shown in Figure A-1 an equilibrium force balance on a particle of the system yields this differential equation of motion.

$$- \frac{(\alpha A)}{g} \frac{\partial^2}{\partial t^2} \left(u + \frac{dy}{dx} \bar{y}\right) + EA \frac{d^2u}{dx^2} = 0 \qquad (1)$$

where:

(αA) = weight per unit length

g = gravity

$\bar{y} = y_2 - y_1$ = distance of center of stiffness from the center of gravity

E = Young's modulus

u = displacement in the x- direction of the center of stiffness from the equilibrium position

$S = E \frac{du}{dx}$

Letting $\quad - \frac{\partial^2}{\partial t^2} (\quad) = w^2 (\quad)$

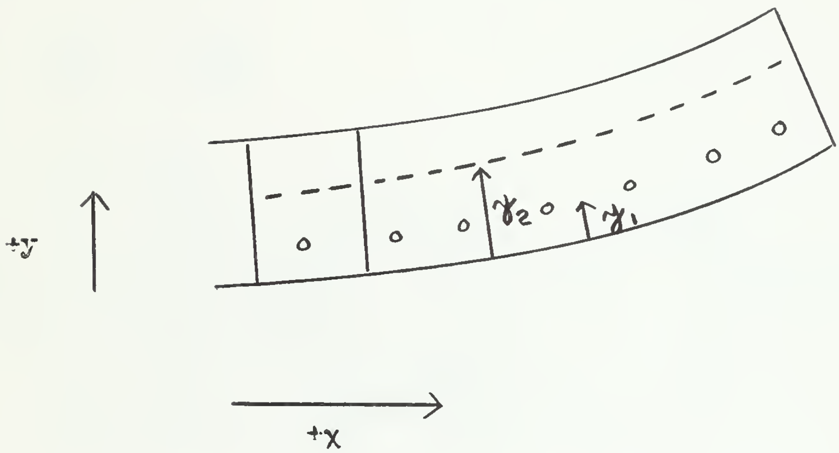
the equation can be rewritten

$$\frac{(\alpha A)}{g} w^2 \left(u + \frac{dy}{dg} \bar{y}\right) + EA \frac{d^2u}{dx^2} = 0 \qquad (1a)$$

A moment balance about the center of gravity of a small particle can be written using the sign convention of Figure A-II.

FIGURE A-1

LONGITUDINAL FORCE BALANCE



y_2 = distance to the center of bending

y_1 = distance to the center of gravity

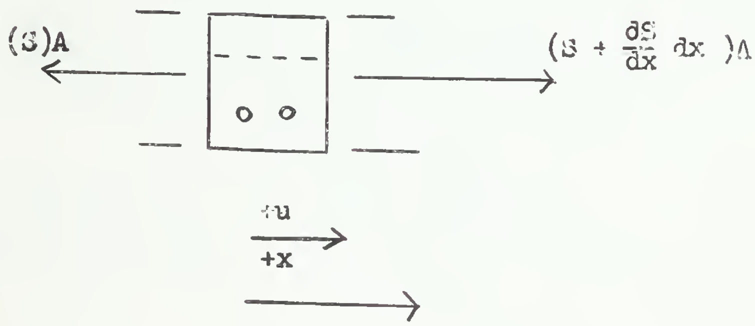
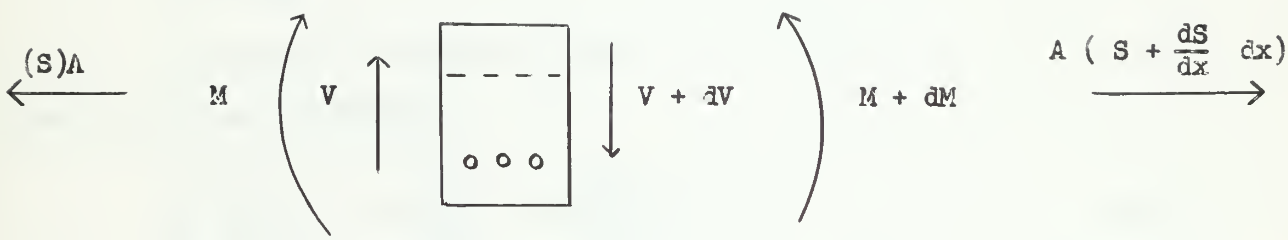
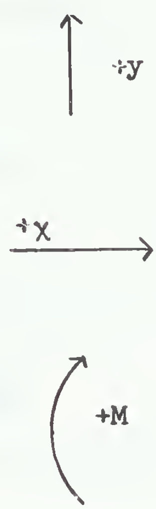


FIGURE A-II

MOMENT BALANCE



SIGN CONVENTION



$$V\,dx + M - (M + dM) + \left(S + \frac{dS}{dx}\,dx - S\right) A\bar{y} = 0 \qquad (2)$$

Let

$$dV = \frac{(\alpha A)}{g} \frac{\partial^2 y}{dt^2}\,dx$$

$$\frac{M}{EI} = -\frac{d^2 y}{dx^2}$$

$$S = E\,\frac{du}{dx}$$

A = area of the cross section

Equation (2) can be rewritten:

$$\frac{\alpha A}{g} \frac{\partial^2 y}{\partial t^2} + EI\,\frac{\partial^4 y}{\partial x^4} + EA\,\frac{\partial^3 u}{\partial x^3}\,\bar{y} = 0 \qquad (2a)$$

Step 2: Assumed Mode Shape and Motion

According to the classical theory of vibration, the equation of transverse motion for a uniform beam free in space is:

$$y = c\left[\text{Cos}\,\lambda x + \left(\frac{\text{Cos}\,\lambda L/2}{\text{Cosh}\,\lambda L/2}\right) \text{Cosh}\,\lambda x\right] \qquad (3)$$

L = length of the beam

x = distance from mid-length

c = arbitrary constant

For the fundamental two-noded mode of vibration:

$$\lambda L = 4.73 = \sqrt[4]{w^2\,\frac{\alpha A}{gEI}}\;L \qquad (3a)$$

Step 3: Longitudinal Motion

Equation 3 is substituted into Equation 2 and the resulting differential equation solved for the longitudinal motion, u. The arbitrary constants are evaluated by assuming no longitudinal motion at mid-length and no stress at the ends.

$$x = 0 \qquad u = 0$$

$$x = L/2 \qquad \frac{du}{dx} = 0$$

The equation for longitudinal motion becomes:

$$u = F_1 \text{ Sin } \frac{x}{\sqrt{D_1}} + F_2 \text{ Sin } \lambda x + F_3 \text{ Sinch } \lambda x \tag{4}$$

where

$$F_1 = - D_1 \lambda D_2 \left[\frac{2 D_1 \lambda^2}{1 - D_1{}^2 \lambda^4}\right] \left[\frac{\text{Cos } \lambda L/2}{\text{Cos } L/2\sqrt{D_1}}\right]$$

$$F_2 = \frac{D_2}{1 - D_1 \lambda^2}$$

$$F_3 = \frac{D_2 D_3}{1 + D_1 \lambda^2}$$

$$D_1 = \frac{gEA}{w^2 (\alpha A)} \tag{5}$$

$$D_2 = \bar{y} c \lambda \tag{6}$$

$$D_3 = - \frac{\text{Cos } \lambda L/2}{\text{Cosh } \lambda L/2} \tag{7}$$

Step 4: Additional Deflection Caused by Longitudinal Forces

In this step the objective is to find the additional vertical deflection caused by the longitudinal motion and forces. Again referring to Figure A-2, the differential equation to be satisfied is:

$$- \frac{\partial M}{\partial x} + \frac{ds}{dx} A\bar{y} = 0 \tag{8}$$

where

$$\frac{M}{EI} = - \frac{\partial^2 y'}{\partial x^2}$$

$$s = E \frac{du}{dx}$$

y' = additional transverse deflection

Equation (8) becomes

$$\frac{d^3 y'}{dx^3} = -\frac{A}{I}\ \bar{y}\ \frac{d^2 u}{dx^2} \tag{9}$$

These boundary conditions are applicable in the solution of equation (9):

$$\frac{d^2 y'}{dx^2} = 0 \qquad \chi = L/2$$

$$\frac{du}{dx} = 0 \qquad x = L/2$$

$$\frac{dy'}{dx} = 0 \qquad x = 0$$

$$u = 0 \qquad x = 0$$

The result of substituting Equation (4) into Equation (9) and solving the differential equation is the equation for the additional deflection of the beam caused by the coupled longitudinal motion. The equation is:

$$y' = \frac{A}{I}\left[F_1\sqrt{D_1}\ \mathrm{Cos}\frac{x}{\sqrt{D_1}} + \frac{F_2}{\lambda}\ \mathrm{Cos}\ \lambda x - \frac{F_3}{\lambda}\ \mathrm{Cosh}\ \lambda x\right] + \text{Constant} \tag{10}$$

A measure of the total additional deflection in the beam is obtained by evaluating Equation (10) at midlength and at the end.

$$Y_1 = y'_{x=0} - y'_{\chi = L/2} \tag{11}$$

$$Y_1 = \frac{A\bar{y}^2}{I} c\ \mathrm{Cos}\ \lambda L/2 \left[\frac{-2\ D^2_1\ \lambda^4}{1 - D_1^{\ 2}\ \lambda^4}\ \frac{\left(\mathrm{Cos}\frac{L/2}{\sqrt{D_1}} - 1\right)}{\mathrm{Cos}\frac{L/2}{\sqrt{D_1}}} + \frac{(\mathrm{Cos}\ \lambda L/2 - 1)}{(1 - D_1\ \lambda^2)\ \mathrm{Cos}\ \lambda L/2}\right.$$

$$\left. + \frac{1}{1 + D_1\ \lambda^2}\ \frac{(\mathrm{Cosh}\ \lambda L/2 - 1)}{\mathrm{Cosh}\ \lambda L/2}\right] \tag{12}$$

<u>Step 5: The Second Approximation to Total Transverse Deflection:</u>

When Equation (10) is then cycled through Steps 3 and 4 and evaluated at the mid-length and end point, the result is a second approximation to additional deflection of the beam.

$$Y_2 = \left(\frac{A}{I}\right)^2 (\bar{y})^4 c \operatorname{Cos} \lambda L/2 \left[\left(\frac{-2 D_1^{\,2} \lambda^4}{1 - D_1^{\,2} \lambda^4} - \frac{4 D_1^{\,2} \lambda^4}{1 - D_1^{\,2} \lambda^4} \right) \left(\frac{\operatorname{Cos} \frac{L/2}{\sqrt{D_1}} - 1}{\operatorname{Cos} \frac{L/2}{\sqrt{D_1}}} \right) \right.$$

$$- \left(\frac{D_1^2 \lambda^4}{1 - D_1^2 \lambda^4} \right) \left(\frac{L/2}{\sqrt{D_1}} \right) \frac{\operatorname{Sin} L/2\sqrt{D_1}}{\left(\operatorname{Cos} \frac{L/2}{\sqrt{D_1}}\right)^2} + \frac{1}{(1 - D_1 \lambda^2)^2} \; \frac{(\operatorname{Cos} \lambda L/2 - 1)}{\operatorname{Cos} \lambda L/2}$$

$$\left. + \frac{1}{(1 + D_1 \lambda^2)^2} \; \frac{(\operatorname{Cosh} \lambda L/2 - 1)}{\operatorname{Cosh} \lambda L/2} \right] \tag{13}$$

<u>Step 6: Calculation of Natural Frequency.</u>

With an assumed frequency and deflection, the Stodola procedure is to calculate the inertia loading and resultant deflection. If the assumed frequency equaled the natural frequency, the resultant deflection would equal the assumed deflection. Mathematically:

$$w_1^{\,2} \, y_1 = w_o^{\,2} \, y_o \tag{14}$$

$$w_1^{\,2} = w_o^{\,2} \, \frac{y_o}{y_1} \tag{15}$$

y_o = assumed deflection

y_1 = resultant deflection

w_o = assumed frequency

w_1 = natural frequency of the beam

When the Stodola equation is applied to the approximate analytical procedure, the equation for determining the natural frequency of a system with a coupling of longitudinal and transverse motion becomes:

$$w_t^{\,2} = w_{ot}^{\,2} \, \frac{Y_o}{Y_o + Y_1 + Y_2}$$

Y_o is Equation (3) evaluated at midlength and at the end

Y_1 is Equation (12)

Y_2 is Equation (13)

w_{ot} = natural transverse frequency of the system without coupling of longitudinal and transverse motion.

w_t = natural frequency of the system with coupling of longitudinal and transverse motion.

Additional information can be obtained by some algebraic manipulation. From Reference (2), the equations for the square of the natural longitudinal frequency and transverse frequency of a uniform beam are respectively:

$$(w_{o\ell})^2 = \left(\frac{\pi}{L}\right)^2 \frac{EgA}{(\alpha A)}$$

$$(w_{ot})^2 = \frac{22.4^2}{L^4} \frac{EgI}{(\alpha A)}$$

Therefore,

$$f^2 = \frac{w_{ot}^{\;2}}{w_{o\ell}^{\;2}} = \frac{22.4^2}{\pi^2} \frac{1}{L^2} \frac{I}{A} \tag{16}$$

Equation (16) gives the very important result that the frequency ratio, f, is independent of mass per unit length.

An examination of the expression $D_1 \lambda^2$ yields another important result. Equations (3a) and (5) show that $D_1 \lambda^2$ is equal to:

$$D_1 \lambda^2 = \left[\frac{gEA}{w_t^2 \, (\alpha A)}\right] \left(\frac{4.73}{L}\right)^2$$

$$= \frac{1}{w_t^2} \left[\frac{gEA}{(\alpha A)} \frac{1}{L^2}\right] (4.73)^2$$

$$= \left(\frac{w_{o\ell}}{w_t}\right)^2 \left(\frac{4.73}{\pi}\right)^2$$

If the change in transverse frequency is small, w_t w_{ot} .

Therefore,

$$D_1 \lambda^2 = \left(\frac{1}{f}\right)^2 \left(\frac{4.73}{\pi}\right)^2$$

Similarly, it can be shown that:

$$\frac{L/2}{\sqrt{D_1}} = f\,\frac{\pi}{2}$$

These manipulations show that Equations (12) and (13) are made up to two factors; one is $\frac{A\bar{y}^2}{I}$ and the other is a complicated expression which is dependent only on frequency ratio.

APPENDIX B

DETAILS OF THE LUMPED MASS SYSTEM

The equations in the beam sequence calculation are derived from the fundamental equation for bending in a beam.

$$EI \frac{d^2y}{dx^2} = M$$

In the case of a beam with the center of gravity and center of stiffness coincident, this equation and its integrals, using the sign convention of Figure B-I, become:

$$EI \frac{d^2y}{dx^2} = M_o + V_o x$$

$$EI \frac{dy}{dx} = M_o x + V_o \frac{x^2}{2} + c_1$$

$$EI\, y = \frac{M_o x^2}{2} + \frac{V_o x^3}{6} + c_1 x + c_2$$

where

V_o = shear at the boundary or beam end

M_o = moment at the boundary or beam end

When these quations are used in a step-by-step procedure, the equations for moment, shear, slope, and deflection when evaluated at an incremental distance to the right of mass 1, shown in Figure B-I, become:

$$M = M_o + V_o \ell$$

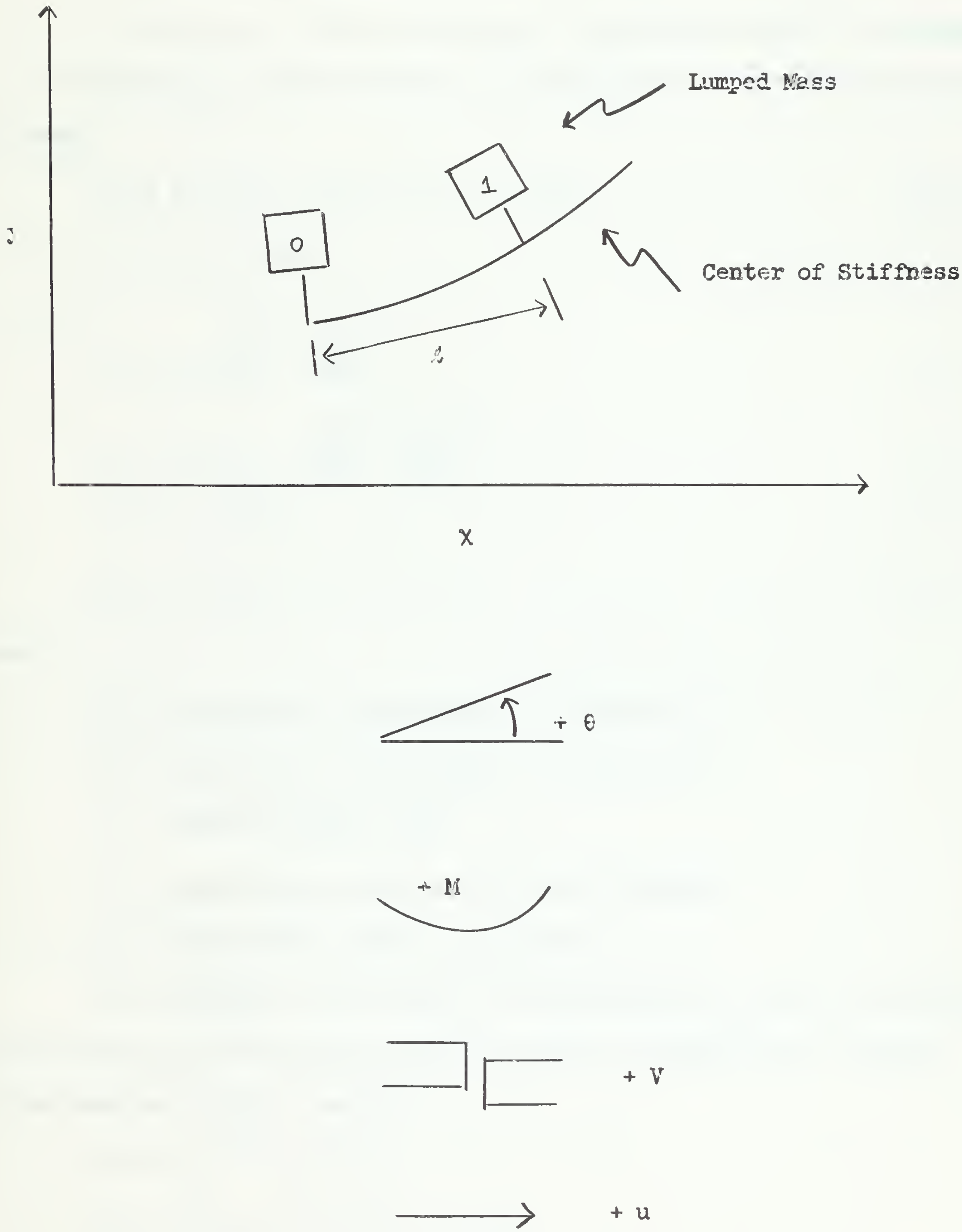
$$V = V_o + M_1 w^2 y_1$$

$$\theta = \theta_o + \frac{M_o \ell}{EI} + \frac{V_o \ell^2}{2EI}$$

$$y = y_o + \theta_o \ell + \frac{M_o \ell^2}{2EI} + \frac{V_o \ell^3}{6EI}$$

FIGURE B-I

SIGN CONVENTION FOR THE LUMPED MASS SYSTEM



where

y_o = deflection at the end

θ_o = slope at the end

In the case of where the center of gravity and center of stiffness are not coincident, these equations are modified and two additional equations are added.

$$M_1 = M_o + V_o \ell + \bar{y}\, m_1 w^2 (u_1 - \bar{y}\, \theta_1) \tag{1}$$

$$V_1 = V_o + m_1 w^2 y_1 \tag{2}$$

$$\theta_1 = \theta_o + \frac{M_o \ell}{EI} + \frac{V_o \ell^2}{2EI} \tag{3}$$

$$y_1 = y_o + \theta_o \ell + \frac{M_o \ell^2}{2EI} + \frac{V_o \ell^3}{6EI} \tag{4}$$

$$u_1 = u_o - \frac{F_o}{K} \tag{5}$$

$$F_1 = F_o + m_1 w^2 (u_1 - \bar{y}\, \theta_1) \tag{6}$$

where

u_o = longitudinal displacement at position zero

u_1 = longitudinal displacement at position one

K = spring constant = $\frac{AE}{\ell}$

F_o = longitudinal force at the end of the beam

F_1 = longitudinal force at position 1

$\bar{y}$ = distance of the center of gravity from the center of stiffness

In the case of a beam vibrating free in space, the conditions at the end of the beam are further specified.

$$F_o = m_o w^2 (u_o - \bar{y}\, \theta_o)$$

$$M_o = F_o \bar{y} = m_o w^2 \bar{y} (u_o - \bar{y}\, \theta_o)$$

$$V_o = m_o w^2 y_o$$

Equations (1) through (6) can be evaluated at other positions along the beam using a table as shown in Appendix C. To start the calculation, let

$$y_o = 1$$

$$u_o = u_o$$

$$\theta_o = \theta_o$$

Then a table is completed for each mass along the system using the results of the previous table to start the next. The end of the last table, which will be the conditions at the mid-point of the beam, will yield three equations:

$$\theta = a_{11} + a_{21}\, u_o + a_{31}\, \theta_o$$

$$V = a_{12} + a_{22}\, u_o + a_{32}\, \theta_o$$

$$u = a_{13} + a_{23}\, u_o + a_{33}\, \theta_o$$

If the proper frequency was used in the calculations, these equations all equal zero. The procedure is to let u and V equal zero, solve for u_o and θ_o and substitute into the θ - equation. When θ equals zero, the proper frequency was used in the calculations.

APPENDIX C

SAMPLE CALCULATIONS OF LUMPED MASS METHOD AT A FREQUENCY RATIO OF .25 WITH ELEVEN MASSES

$A = 1 \text{ in.}^2$

$I = 1/12 \text{ in.}^4$

$\alpha = .284 \text{ lbs./in.}^3$

$L = 8.24 \text{ in.}$

$f = .25$

$\bar{y} = 1/4 \text{ in.}$

$\ell = .824 \text{ in.}$

$m = .60666 \times 10^{-3} \text{ (lbs.-sec.}^2\text{/in.)}$

$w_{ot}^2 = 3.69543 \times 10^8 \text{ (radians/sec.)}^2$

$\frac{\ell}{EI} = .03295 \times 10^{-5} \text{ (1/lbs.-in)}$

$\frac{\ell^2}{2EI} = .01357 \times 10^{-5} \text{ (1/lbs.)}$

$\frac{\ell^3}{6EI} = .00372 \times 10^{-5} \text{ (in./lbs.)}$

$\frac{1}{K} = .00274 \times 10^{-5} \text{ (in./lbs.)}$

assume $w = .975\ w_{ot}$

Therefore,

$mw^2 = 2.13120 \times 10^5$

$F_o = 1.06560 \times 10^5\ u_o - .26640 \times 10^5\ \theta_o$

$M_o = .26640 \times 10^5\ u_o - .06660 \times 10^5\ \theta_o$

$V_o = 1.0650 \times 10^5\ y_o$

The five-page tabular calculation to reach the center of the beam is on the following pages.

	θ			Y			V			M		
	y_o	u_o	θ_o	y_o	u_o	θ_o	y_o	u_o	θ_o	y_o	u_o	θ_o
Y				1.0000	0	0						
θ	0	0	1.0000									
ℓ				0	0	.82400						
M										0	.26640	-.06660
ℓ/EI	0	.00877	-.00219									
ℓ^2/2EI				0	.00361	-.00090						
V							1.06560	0	0			
ℓ										.87805	0	0
ℓ^2/2EI	.01446	0	0									
ℓ^3/6EI				.00396								
mw^2y							2.13963	.00769	1.75419			
F'y										-.00192	.53005	-.13246
Total	.01446	.00877	.99781	1.00396	.00361	.82310	3.20523	.00769	1.75419	.87613	.79645	-.19906

	u			F		
	y_o	u_o	θ_o	y_o	u_o	θ_o
u	0	1.000	0			
F				0	1.06560	-.26640
-1/K	0	-.00291	.00077			
F'				-.00769	2.12033	-.52985
Total	0	.99709	.00073	-.00769	3.18593	-.79625

	y_o	u_o	θ_o
u	0	.99709	.00073
$-y\theta$	-.00361	-.00219	-.24945
$u-y\theta$	-.00361	.99490	-.24872
F'	-.00769	2.12033	-.52985

$$F' = mw^2 (u - \bar{y}\theta)$$

	θ			Y			V			M		
	y_o	u_o	θ_o	y_o	u_o	θ_o	y_o	u_o	θ_o	y_o	u_o	θ_o
Y				1.00396	.00361	.82310						
θ	.01446	.00877	.99781									
ℓ				.01191	.00722	.82219						
M										.87613	.79645	¯.19906
ℓ/EI	.02886	.02624	¯.00655									
$\ell^2/2EI$				.01188	.01080	¯.00270						
V							3.20523	.00769	1.75419			
ℓ										2.6440	.00633	1.44545
$\ell^2/2EI$	.04349	.00010	.02380									
$\ell^3/6EI$				.01192	.00002	.00652						
mw^2y							2.21574	.04614	3.51458			
$F'\bar{y}$										¯.01155	.52193	¯.13365
Total	.08681	.03511	1.01506	1.03967	.02165	1.64911	5.42097	.05383	5.26877	3.50568	1.32471	1.11274

	u			F		
	y_o	u_o	θ_o	y_o	u_o	θ_o
u	0	.99709	.00073			
F				¯.00769	3.18593	¯.79625
-1/K	¯.00002	¯.00872	¯.00218			
F'				¯.04620	2.08772	¯.53461
Total	.00002	.98837	.00291	¯.05389	5.27365	¯1.33087

u	.00002	.98837	.00291
$-\bar{y}\theta$	¯.02170	¯.00877	¯.25376
$u-\bar{y}\theta$	¯.02168	.97960	¯.25085
F'	¯.04620	2.08772	¯.53461

	θ			Y			V			M		
	y_o	u_o	θ_o	y_o	u_o	θ_o	y_o	u_o	θ_o	y_o	u_o	θ_o
Y				1.03967	.02165	1.14911						
θ	.08681	.03511	1.01506									
ℓ				.07153	.02893	.83640						
M										3.50568	1.32471	1.11274
ℓ/EI	.11551	.04364	..03666									
$\ell^2/2EI$				.04756	.01797	.01509						
V							5.42097	.05383	5.26877			
ℓ										4.46687	.04435	4.34146
$\ell^2/2EI$	.07356	.00073	..071119									
$\ell^3/6EI$				.02016	.00020	.01959						
mw^2y							2.51251	.14652	5.37102			
$F'\bar{y}$										$\bar{.}03671$	.50832	$\bar{.}14612$
Total	.27588	.07948	1.12321	1.17892	.06875	2.52019	7.93348	.20035	10.63971	7.93584	1.87738	5.30808

	u			F		
	y_o	u_o	θ_o	y_o	u_o	θ_o
u	.00002	.98837	.00241			
F				$\bar{.}05389$	5.27365	$\bar{1}.33087$
-1/K	.00014	$\bar{.}01444$	.00364			
F'				$\bar{.}14686$	2.03329	$\bar{.}58448$
Total	.00016	.97393	.00655	$\bar{.}20075$	7.30694	$\bar{1}.91535$

	y_o	u_o	θ_o
u	.00016	.97393	.00655
$-\bar{y}\theta$	$\bar{.}06897$	.01987	$\bar{.}28080$
$u-\bar{y}\theta$	$\bar{.}06881$	.95406	$\bar{.}27425$
F'	$\bar{.}14684$	2.03329	$\bar{.}58448$

	θ y_o	θ u_o	θ θ_o	Y y_o	Y u_o	Y θ_o	V y_o	V u_o	V θ_o	M y_o	M u_o	M θ_o
Y				1.17892	.06875	2.52016						
θ	.27588	.07948	1.12321									
ℓ				.22732	.06549	.92552						
M										7.93584	1.87738	5.30808
ℓ/EI	.26148	.06185	.17490									
$\ell^2/2EI$				.10768	.02547	.07206						
V							7.93340	.20035	10.63979			
ℓ										6.53712	.16508	8.76718
$\ell^2/2EI$	.10765	.00271	.14438									
$\ell^3/6EI$				.02951	.00074	.03958						
$m\omega^2 y$							3.28935	.34195	7.58136			
$F'y$										-.08553	.48705	-.18585
Total	.64501	.14404	1.44249	1.54343	.16045	3.55732	11.22275	.55030	18.22115	14.38743	2.52951	13.88941

	u y_o	u u_o	u θ_o	F y_o	F u_o	F θ_o
u	.00016	.97393	.00655			
F				-.20075	7.30694	-1.91535
-1/K	.00055	-.02002	.00524			
F'				-.34214	1.95622	-.74342
Total	.00071	.95391	.01179	-.54289	9.26316	-2.65877

u	.00071	.95391	.01179
$-\bar{y}\theta$	-.16125	-.03601	-.36062
$u-\bar{y}\theta$	-.16054	.91790	-.34883
F'	-.34214	1.95622	-.74342

	θ			Y			V			M		
	y_o	u_o	θ_o	y_o	u_o	θ_o	y_o	u_o	θ_o	y_o	u_o	θ_o
Y				1.54343	1.6045	3.55732						
θ	.64501	.14404	1.44249									
ℓ				.53148	.11868	1.18861						
M										14.38743	2.52951	13.88941
ℓ/EI	.47406	.08334	.45765									
$\ell^2/2EI$				.19523	.03432	.18847						
V							11.22275	.55030	18.22115			
ℓ												
$\ell^2/2EI$	.15229	.00746	.24726									
$\ell^3/6EI$				.04174	.00204	.06778						
mw^2y							2.46353	.33618	5.33032			
$F'\bar{y}$												
Total	1.27136	.23484	2.14740	2.31188	.31549	5.00218	13.68628	.88648	23.55147			

	u			F		
	y_o	u_o	θ_o	y_o	u_o	θ_o
u	.00071	.95391	.01179			
F				.54285	9.26316	-2.65877
-1/K	.00148	-.02538	.00728			
F'						
Total	.00219	.92853	.01907			

Note: (1) All calculations are not shown on the last page, since only θ, V, and u are needed to satisfy boundary conditions at the mid-length.

The last page of the tabular calculation yields three equations which would all equal zero if the right frequency was assumed at the beginning of the calculation.

$$\theta = 1.27136\, y_o + .23484\, u_o + 2.14740\, \theta_o$$

$$V = 13.68628\, y_o + .88648\, u_o + 23.55147\, \theta_o$$

$$u = .00219\, y_o + .92853\, u_o + .01907\, \theta_o$$

let $y_o = 1$

$u = 0$

$V = 0$

Solve for u_o and θ_o and substitute into the θ equation.

$$u_o = .00958$$

$$\theta_o = -.58148$$

$$\theta = .02493$$

When the assumed frequency is .95 w_{ot}, and the five-page tabular calculation repeated,

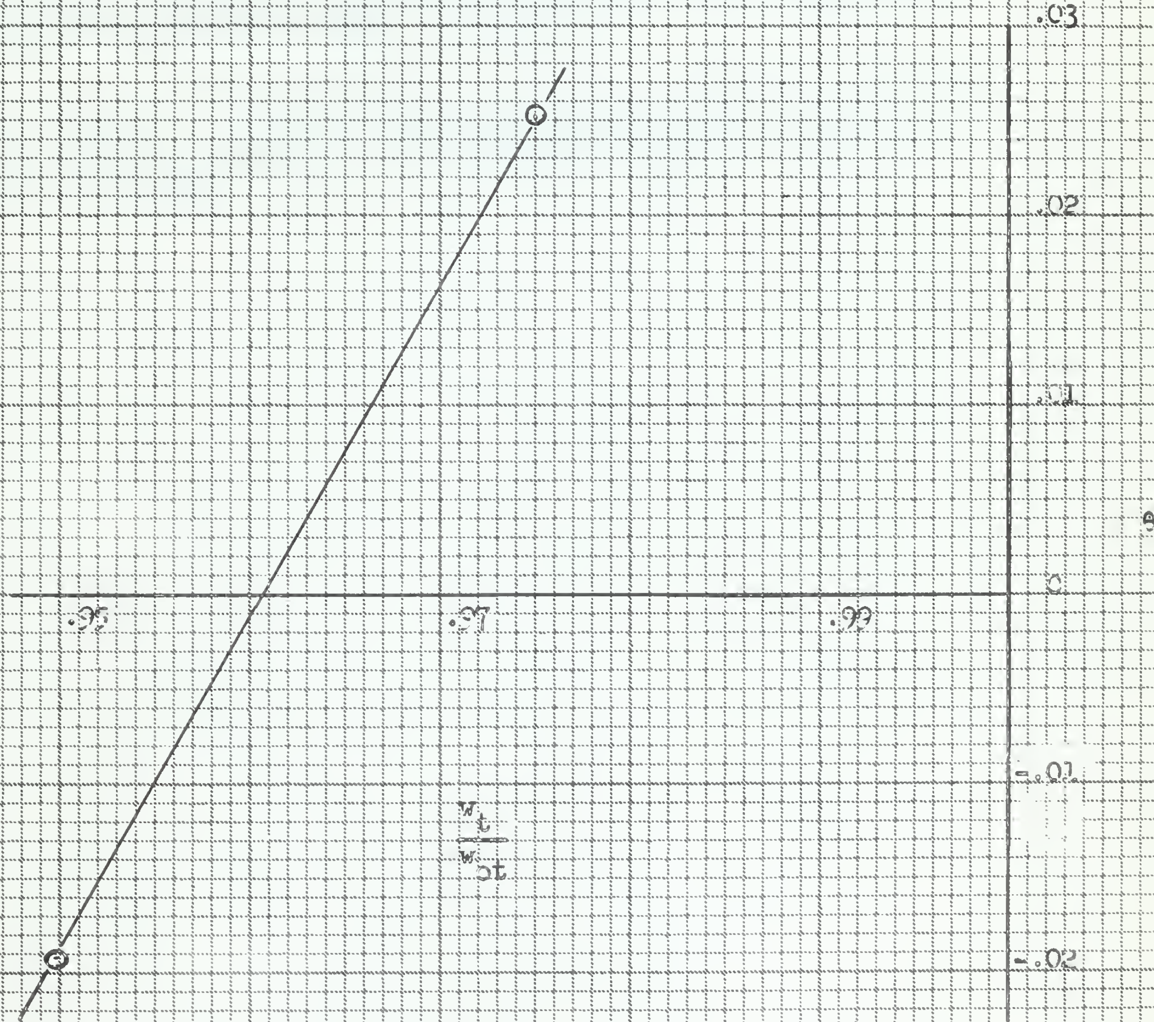
$$\theta = -\ .01893$$

The result is plotted in Figure C-1, and thus the natural transverse frequency with offset center of gravity is:

$$w = .961\, w_{ot}$$

FIGURE C-I

SLOPE AT MID-LENGTH VERSUS ASSUMED NATURAL FREQUENCY



BIBLIOGRAPHY

1. Reed, F. W., and Bradshaw, R. T., Ship Hull Vibrations - I, A Bibliography, Contract No. Nobs 77150, Bureau of Ships, Navy Department, 1960.

2. Den Hartog, J. P., Mechanical Vibrations, McGraw-Hill Book Company, New York, 1947.

3. McGoldrick, R. T., Ship Vibration, David Taylor Model Basin Report 1451, December, 1960, page A-4.

4. Gesswein, G. H., and Moses, F., Calculated Modes and Frequencies of Hull Vibration of USS GEORGE WASHINGTON (SSBN 598), David Taylor Model Basin Report 1464, November, 1960, pp. 11-13.

5. Duncan, W. J., "A Critical Examination of the Representation of Massive and Elastic Bodies by Systems of Rigid Masses Elastically Connected", Quarterly Journal of Mechanics and Applied Mathematics, Volume 5, March 1952, pp. 97-108.